

ON EXT-INDICES OF RING EXTENSIONS

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ABSTRACT. In this paper we are concerned with the finiteness property of Ext-indices of several ring extensions. In this direction, we introduce some conjectures and discuss the relationship of them. Also we give affirmative answers to these conjectures in some special cases. Furthermore, we prove that the trivial extension of an Artinian local ring by its residue class field is always of finite Ext-index and we show that the Auslander-Reiten conjecture is true for this type of rings.

1. INTRODUCTION

Throughout the paper, all rings are assumed to be commutative Noetherian rings with unity.

Let R be a ring. According to [2], given nonzero R -modules M and N , we define $p^R(M, N)$ by the following equality:

$$p^R(M, N) = \sup\{i \in \mathbb{N} \mid \text{Ext}_R^i(M, N) \neq 0\} (\leq \infty).$$

And we define the Ext-index of the ring R , denoted by $\text{Ext-index}(R)$, to be the supremum of finite values of $p^R(M, N)$ for finitely generated R -modules M and N , i.e.

$$\text{Ext-index}(R) = \sup\{p^R(M, N) \mid M \text{ and } N \text{ are finitely generated } R\text{-modules with } p^R(M, N) < \infty\}.$$

Definition 1.1. We say that the ring R is **of finite Ext-index** if it satisfies $\text{Ext-index}(R) < \infty$. Following the paper [7], we often call R an AB ring if it is a Gorenstein local ring of finite Ext-index.

Recall that the following rings are examples known to be of finite Ext-index:

Complete intersections ([7, Corollary 3.5]), Golod rings ([9, Proposition 1.4]), Gorenstein local rings with minimal multiplicity ([7, Theorem 3.6]), and Gorenstein local rings with codimension at most 4 ([12, Theorem 3.4]).

Note also from [9] that there exists an example of an Artinian Gorenstein local ring which is not AB.

In this paper we are mainly concerned with the finiteness property of Ext-indices of several ring extensions. This is motivated by the following conjectures, all of which seem to be open. (See also [4].)

2000 *Mathematics Subject Classification.* 13C10, 13D07, 16E30.

Key words and phrases. AB ring, trivial extension, Cohen-Macaulay ring, Auslander-Reiten conjecture.

Conjecture (L) : Let R be a ring and let $\mathfrak{p} \in \text{Spec}(R)$. If R is of finite Ext-index, then so would be the localization $R_{\mathfrak{p}}$.

Conjecture (E) : Let R be an algebra over a field k and let ℓ be a finitely generated extension field of k . If R is of finite Ext-index, then so would be the ring $R \otimes_k \ell$.

Conjecture (P) : Let R be a ring. If R is of finite Ext-index, then so would be the polynomial ring $R[x]$.

In Section 2, after making some preliminaries, we discuss the relationship among these conjectures (Proposition 2.8). We shall also give some of the obvious cases for the above conjectures.

In Section 3, we are interested in the trivial extension $R(k)$ of an Artinian local ring R with its residue class field k . Surprisingly enough, we prove that $R(k)$ is always of finite Ext-index (Corollary 3.4). Furthermore we can show that the Auslander-Reiten conjecture is true for the rings of this type (Corollary 3.6).

In Section 4, we are interested in how the finiteness of Ext-index is preserved by a base field extension for algebras. To be precise let R be a finite dimensional algebra over a field k and we consider a transcendental extension $k(x)$ of k . We show under a mild assumption that if R is of finite Ext-index then so is the extended ring $R \otimes_k k(x)$. See Theorem 4.2 for the detail. We also give some variants of this theorem in Theorems 4.4 and 4.6.

For unexplained notation and terminologies in the paper, see the books [3], [5], [10] and [13].

2. PRELIMINARIES

We recall some of the basic facts concerning the Ext-indices.

Lemma 2.1.

- (1) *Let $R \rightarrow S$ be a faithfully flat ring homomorphism. Then the inequality $\text{Ext-index}(R) \leq \text{Ext-index}(S)$ holds. In particular, if S is of finite Ext-index, then so is R .*
- (2) *Let $R = R_1 \times R_2$ be a product of rings. Then we have an equality*

$$\text{Ext-index}(R) = \sup\{\text{Ext-index}(R_1), \text{Ext-index}(R_2)\}.$$

In particular, R is of finite Ext-index if and only if so are the both of R_1 and R_2 .

- (3) *Let x be a non-zero divisor of R . Then the inequality $\text{Ext-index}(R/xR) \leq \text{Ext-index}(R) - 1$ holds. In particular, if R is of finite Ext-index, then so is R/xR .*
- (4) *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring with dualizing module. And let x be a non-zero divisor of R that belongs to \mathfrak{m} . If R/xR is of finite Ext-index, then so is R .*

Proof. (1) and (2) are proved straightforward only from the definition. See [4, Proposition 3.5] for (1) and [4, Proposition 3.3] for (2). For (3) and (4), refer to [7, Proposition 3.3(1)], [4, Lemma 3.4] and [4, Propositions 4.2]. \square

In the following we give an obvious case of Conjecture (L). In the lemma, $\text{Max}(R)$ (resp. $\text{Min}(R)$) denotes the set of all maximal (resp. minimal prime) ideals of R .

Lemma 2.2. *Let $\mathfrak{m} \in \text{Max}(R) \cap \text{Min}(R)$. Then we have*

$$\text{Ext-index}(R_{\mathfrak{m}}) \leq \text{Ext-index}(R).$$

In particular, if R is of finite Ext-index, then so is $R_{\mathfrak{m}}$ for each $\mathfrak{m} \in \text{Max}(R) \cap \text{Min}(R)$.

Proof. Let $(0) = Q_1 \cap Q_2$ be an irredundant primary decomposition, where Q_1 is an \mathfrak{m} -primary component and Q_2 is the intersection of the components belonging to other primes. Since \mathfrak{m} is a maximal ideal, we have $Q_1 + Q_2 = R$, hence $R \cong R/Q_1 \times R/Q_2$. Noting that $(0)_{\mathfrak{m}} = Q_{1\mathfrak{m}}$, we see that $R_{\mathfrak{m}} \cong (R/Q_1)_{\mathfrak{m}} \cong R/Q_1$. Therefore the lemma follows from Lemma 2.1(2). \square

Corollary 2.3. *If R is an Artinian ring of finite Ext-index, then so is $R_{\mathfrak{p}}$ for every prime ideal \mathfrak{p} of R .*

Lemma 2.4. *Let R be a Cohen-Macaulay ring of finite Ext-index. Then $R_{\mathfrak{m}}$ is of finite Ext-index for every maximal ideal \mathfrak{m} of R .*

Proof. Let $d = \text{ht}(\mathfrak{m}) \geq 0$. We can take a regular sequence $x_1, \dots, x_d \in \mathfrak{m}$ on R . Then $R/(x_1, \dots, x_d)R$ is of finite Ext-index by 2.1(3). And thus $(R/(x_1, \dots, x_d)R)_{\mathfrak{m}}$ is an Artinian ring of finite Ext-index by Lemma 2.2. Apply 2.1(4) to this ring that is isomorphic to $\widehat{(R_{\mathfrak{m}})}/(x_1, \dots, x_d)\widehat{(R_{\mathfrak{m}})}$, and we see that $\widehat{(R_{\mathfrak{m}})}$ is of finite Ext-index. Finally, by virtue of 2.1(1), we have the finiteness of Ext-index of $R_{\mathfrak{m}}$. \square

The following is a corollary of the proof above.

Corollary 2.5. *Let R be a Cohen-Macaulay local ring of finite Ext-index. Then the completion \widehat{R} is also of finite Ext-index.*

Lemma 2.6. *Let R be a Gorenstein ring of finite Ext-index and suppose that $\dim(R) < \infty$. Then the equality $\text{Ext-index}(R) = \dim(R)$ holds.*

Before proving this, we should remark that if R is an AB ring then the equality was shown in [7, Proposition 3.2]. Also this has been proved by Mori [11, Corollary 3.3] including non-commutative cases. We give below the proof for the convenience of the reader.

Proof. Let \mathfrak{m} be a maximal ideal of R . By 2.4, we know that $R_{\mathfrak{m}}$ is an AB ring. It follows from the above mentioned result of [7] that $\text{Ext-index}(R_{\mathfrak{m}}) = \text{ht}(\mathfrak{m}) \leq \dim(R)$.

Now let M and N be finitely generated R -modules with $\text{Ext}_R^i(M, N) = 0$ for $i \gg 0$. Then we have obviously $\text{Ext}_{R_{\mathfrak{m}}}^i(M_{\mathfrak{m}}, N_{\mathfrak{m}}) = 0$ for $i \gg 0$, hence the equality

holds for all $i > \dim(R) \geq \text{Ext-index}(R_{\mathfrak{m}})$. This is true for any maximal ideal \mathfrak{m} . Hence we have $\text{Ext}_R^i(M, N) = 0$ for $i > \dim(R)$. Therefore, it is concluded that $\text{Ext-index}(R) \leq \dim(R)$.

On the other hand, since $\dim(R) = \text{id}(R)$, we can find a finitely generated R -module L such that $\text{Ext}_R^d(L, R) \neq 0$, and $\text{Ext}_R^i(L, R) = 0$ for all $i > d = \dim(R)$. This implies that $\dim(R) \leq \text{Ext-index}(R)$. \square

Lemma 2.7. *Let R be a Gorenstein ring of finite Krull dimension and suppose $R_{\mathfrak{m}}$ is of finite Ext-index for each $\mathfrak{m} \in \text{Max}(R)$. Then R is of finite Ext-index.*

Proof. In fact, completely as in the same way as in the proof of 2.6 we can prove the following inequalities:

$$\text{Ext-index}(R) \leq \sup\{\text{Ext-index}(R_{\mathfrak{m}}) \mid \mathfrak{m} \in \text{Max}(R)\} \leq \dim(R) < \infty.$$

\square

We observe the relationship among Conjectures (L), (E) and (P) introduced in Section 1.

Proposition 2.8.

- (1) *Suppose that Conjecture (P) is true for all Cohen-Macaulay local rings R of dimension one. Then Conjecture (L) is true for all Cohen-Macaulay local rings R of any dimension with dualizing module.*
- (2) *Suppose that Conjecture (P) is true for a k -algebra R . Then Conjecture (E) is true for R and for all simple algebraic extensions ℓ of k .*
- (3) *Suppose that Conjectures (L) and (E) are true for all Gorenstein rings containing field. Then Conjecture (P) is true for all Gorenstein rings of finite Krull dimension that contain fields.*

Proof. (1) Suppose (P) holds for all Cohen-Macaulay local rings of dimension one. Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring with dualizing module and let $\mathfrak{p} \in \text{Spec}(R)$, and assume that R is of finite Ext-index. To prove that $R_{\mathfrak{p}}$ is of finite Ext-index, by induction on $\text{ht}(\mathfrak{m}/\mathfrak{p})$, we may assume that $\text{ht}(\mathfrak{m}/\mathfrak{p}) = 1$. Take a maximal regular sequence $\{x_1, \dots, x_h\}$ in \mathfrak{p} (so that $h = \text{ht}(\mathfrak{p})$), and consider the residue ring $\overline{R} = R/(x_1, \dots, x_h)$. By virtue of Lemma 2.1(3) and (4), replacing R by \overline{R} , we may assume that R is of one dimension and $\mathfrak{p} \in \text{Min}(R)$. Then take a non-zero divisor $a \in \mathfrak{m}$. Since $R[x]$ is of finite Ext-index, it follows from Lemma 2.1(3) that $R_a \cong R[x]/(ax - 1)$ is also of finite Ext-index. Since R_a is Artinian and $a \notin \mathfrak{p}$, the localization $R_{\mathfrak{p}}$ is of finite Ext-index as well, by Corollary 2.3.

(2) Let R be a k -algebra of finite Ext-index. Assume $\ell = k(\alpha)$ is a simple algebraic extension of k . Let $f(x)$ be the minimal polynomial of α over k . Then we have $R \otimes_k \ell \cong R[x]/(f(x))$. Since we are assuming that $R[x]$ is of finite Ext-index, $R[x]/(f(x))$ is of finite Ext-index as well, by Lemma 2.1(3).

(3) Let R be a Gorenstein ring of finite Ext-index that contains a field. To prove that the polynomial ring $R[x]$ is of finite Ext-index, we only have to show that $R[x]_{\mathfrak{M}}$ is so for each maximal ideal \mathfrak{M} of $R[x]$. (See Lemma 2.7.) Set $\mathfrak{p} = \mathfrak{M} \cap R$

and we have that $R_{\mathfrak{p}}$ is of finite Ext-index by the assumption that (L) is true for R . Replacing R with $R_{\mathfrak{p}}$, we may assume that (R, \mathfrak{m}) is a Gorenstein local ring of finite Ext-index and that $\mathfrak{M} \cap R = \mathfrak{m}$. By virtue of Lemma 2.1(1) and Corollary 2.5, we may also assume that R is a complete local ring. Since R contains a field, R has a coefficient field k . Then it is obvious that there is an irreducible polynomial $f(x) (\neq 0) \in k[x]$ with $\mathfrak{M} = (\mathfrak{m}, f(x))R[x]$. By Lemma 2.1(4), the finiteness of Ext-index for $R[x]_{\mathfrak{M}}$ follows from that for $(R[x]/(f(x)))_{\mathfrak{M}}$. But the last ring is a localization of $R \otimes_k k[x]/(f(x))$, which is of finite Ext-index by the validity of (E) and (L). \square

As to Conjecture (P) we give an affirmative answer in a special case.

Proposition 2.9. *Let R be an Artinian Gorenstein ring of finite Ext-index. Assume that every residue class field of R is algebraically closed. Then the polynomial ring $R[x_1, \dots, x_n]$ is also of finite Ext-index.*

Proof. By Lemma 2.7, it is enough to prove that $R[x_1, \dots, x_n]_{\mathfrak{M}}$ is of finite Ext-index for every maximal ideal \mathfrak{M} of $R[x_1, \dots, x_n]$. Since R is Artinian, we see that $\mathfrak{M} \cap R = \mathfrak{m}$ is a maximal ideal of R and R/\mathfrak{m} is an algebraically closed field. Therefore, by Hilbert's Nullstellensatz, there are elements $r_1, \dots, r_n \in R$ with $\mathfrak{M} = (\mathfrak{m}, x_1 - r_1, \dots, x_n - r_n)R[x_1, \dots, x_n]$. Since $R_{\mathfrak{m}} \cong R[x_1, \dots, x_n]_{\mathfrak{M}}/(x_1 - r_1, \dots, x_n - r_n)R[x_1, \dots, x_n]_{\mathfrak{M}}$ is of finite Ext-index by Corollary 2.3 and since $\{x_1 - r_1, \dots, x_n - r_n\}$ is a regular sequence contained in the Jacobson radical of $R[x_1, \dots, x_n]_{\mathfrak{M}}$, it follows from Lemma 2.1(4) that $R[x_1, \dots, x_n]_{\mathfrak{M}}$ is of finite Ext-index. \square

3. TRIVIAL EXTENSIONS

Let M be an R -module. Recall that the trivial extension $R(M)$ of R by M is defined to be $R \oplus M$ as an underlying R -module that is equipped with ring structure by defining the multiplication by

$$(r, m) \cdot (r', m') = (rr', rm' + r'm).$$

There are ring homomorphisms $\rho : R \longrightarrow R(M)$ with $\rho(r) = (r, 0)$ and $\pi : R(M) \longrightarrow R$ with $\pi(r, m) = r$. Note that $\pi \cdot \rho$ is the identity mapping on R .

In this section we are mainly concerned with the trivial extension $R(k)$ of the local ring (R, \mathfrak{m}, k) by the residue class field k . We prove the following theorem as a main result of this section.

Theorem 3.1. *Let (R, \mathfrak{m}, k) be a local ring and M, N be nonzero non-free finitely generated $R(k)$ -modules. Then $\text{Tor}_n^{R(k)}(M, N) \neq 0$ for all $n \geq 3$.*

The following is a key to prove the theorem. Notice that any R -module can be regarded as an $R(k)$ -module through $\pi : R(k) \rightarrow R$.

Lemma 3.2. *Let (R, \mathfrak{m}, k) be a local ring. Then for R -modules M and N and for an integer $n \geq 1$ we have an isomorphism*

$$\text{Tor}_n^{R(k)}(M, N) \cong \text{Tor}_n^R(M, N) \oplus \coprod_{i+j=n-1} \text{Tor}_i^{R(k)}(M, k) \otimes_k \text{Tor}_j^R(k, N).$$

Proof. Set $A = R(k)$, $x = (0, 1) \in A$ and let \mathfrak{n} be the maximal ideal of A . Note that $\mathfrak{n} = (0 :_A x)$ holds and there is an isomorphism $R \cong A/Ax$ as a ring. Now consider the short exact sequence of A -modules:

$$0 \longrightarrow k \xrightarrow{x} A \longrightarrow R \longrightarrow 0.$$

This induces the triangle

$$R \otimes_A^{\mathbf{L}} k \longrightarrow R \otimes_A^{\mathbf{L}} A \xrightarrow{\pi} R \otimes_A^{\mathbf{L}} R \longrightarrow R \otimes_A^{\mathbf{L}} k[1],$$

which is actually a triangle in the derived category $D^-(R, A)$ of right bounded chain complexes of (R, A) -bimodules. Consider the natural augmentation $\epsilon : R \otimes_A^{\mathbf{L}} R \longrightarrow H_0(R \otimes_A^{\mathbf{L}} R)$ and we have the following commutative diagram in $D^-(R, A)$:

$$\begin{array}{ccc} R \otimes_A^{\mathbf{L}} A & \xrightarrow{\pi} & R \otimes_A^{\mathbf{L}} R \\ \cong \downarrow & & \downarrow \epsilon \\ R & \xrightarrow{\cong} & H_0(R \otimes_A^{\mathbf{L}} R). \end{array}$$

Thus π has a left inverse ϵ . Therefore the triangle splits off and it gives an isomorphism in $D^-(R, A)$:

$$(*) \quad R \otimes_A^{\mathbf{L}} R \cong R \oplus (R \otimes_A^{\mathbf{L}} k)[1].$$

Note that the natural ring homomorphism $\rho : R \rightarrow A$ induces the forgetting functor $D^-(R, A) \rightarrow D^-(R, R)$, and through this functor we can regard $(*)$ as an isomorphism in $D^-(R, R)$. Now apply the functors $M \otimes_R^{\mathbf{L}} -$ from the left and $- \otimes_R^{\mathbf{L}} N$ from the right to the isomorphism $(*)$, and as a consequence we have an isomorphism in $D^-(R, R)$:

$$M \otimes_A^{\mathbf{L}} N \cong (M \otimes_R^{\mathbf{L}} N) \oplus ((M \otimes_A^{\mathbf{L}} k) \otimes_k^{\mathbf{L}} (k \otimes_R^{\mathbf{L}} N)) [1].$$

The lemma follows by taking the homology modules of the both ends. \square

Remark 3.3. Let S be a local ring with residue class field ℓ and let M, N be S -modules such that $\ell_S(\text{Tor}_n^S(M, N)) < \infty$ for all n . Then we can consider the generating function $P_{M,N}^S(t)$ defined by the equality

$$P_{M,N}^S(t) = \sum_{n \geq 0} \ell_S(\text{Tor}_n^S(M, N)) t^n.$$

Recall that the Poincaré series $P_M^S(t)$ of M is defined to be $P_{\ell, M}^S(t)$ and the Poincaré series $P_\ell^S(t)$ is denoted simply by $P_S(t)$.

Note that by the previous lemma we can show the equality

$$P_{M,N}^{R(k)}(t) = P_{M,N}^R(t) + P_M^{R(k)}(t) P_N^R(t) t,$$

if M and N are finitely generated R -modules with $\ell_{R(k)}(\text{Tor}_n^{R(k)}(M, N)) < \infty$ for all n . Applying this to $M = N = k$, we have

$$P_{R(k)}(t) = P_R(t)(1 - P_R(t) t)^{-1},$$

which is a special case of a theorem of Gulliksen [6, Theorem 2].

Now we proceed to the proof of the theorem.

Proof of Theorem 3.1. We use the same notation as in the proof of Lemma 3.2. Suppose $\text{Tor}_n^A(M, N) = 0$ for some $n \geq 3$. Replacing M and N with their first syzygies, we may assume that $\text{Tor}_n^A(M, N) = 0$ for some $n \geq 1$ and that $xM = 0$ and $xN = 0$, since $M \subseteq \mathfrak{n}F$ and $N \subseteq \mathfrak{n}G$ for some free A -modules F and G . Thus we may assume M and N are modules over R through the identification $R \cong A/Ax$. Then by Lemma 3.2, the equality $\text{Tor}_{n-1}^A(M, k) \otimes_k (N \otimes_R k) = 0$ holds. Since $N \otimes_R k \neq 0$, we see $\text{Tor}_{n-1}^A(M, k) = 0$. This implies that M has finite projective dimension as an A -module. But $\text{depth}(A) = 0$ and by Auslander-Buchsbaum formula, M is a free A -module. This is a contradiction. \square

As applications of Theorem 3.1 we can show the following corollaries.

Corollary 3.4. *Let (R, \mathfrak{m}, k) be an Artinian local ring and let M and N be finitely generated modules over the trivial extension $R(k)$. If $\text{Ext}_{R(k)}^i(M, N) = 0$ for some integer $i \geq 3$, then either M is $R(k)$ -free or N is $R(k)$ -injective.*

In particular, the equality $\text{Ext-index}(R(k)) = 0$ holds.

Proof. Taking the Matlis dual which we denote by $(\)^\vee$, we have $\text{Tor}_i^{R(k)}(M, N^\vee) = 0$ for an integer $i \geq 3$. Hence by Theorem 3.1, one of M and N^\vee is $R(k)$ -free. \square

Corollary 3.5. *Let (R, \mathfrak{m}, k) be an Artinian local ring. And let $E = E_{R(k)}(k)$ be the injective envelope of the $R(k)$ -module k . Then $\text{Ext}_{R(k)}^i(E, R(k)) \neq 0$ for all $i \geq 3$.*

Proof. Otherwise, it follows from Corollary 3.4 that E is $R(k)$ -free or $R(k)$ is $R(k)$ -injective. In either case $R(k)$ must be a Gorenstein ring. However, since the socle dimension of $R(k)$ is bigger than that of R by 1, there is no chance for $R(k)$ to be Gorenstein. \square

Corollary 3.6 (Auslander-Reiten conjecture for $R(k)$). *Let (R, \mathfrak{m}, k) be an Artinian local ring and let M be a finitely generated module over $R(k)$. Suppose that $\text{Ext}_{R(k)}^i(M, M \oplus R(k)) = 0$ for all $i > 0$ (or more weakly, for some integer $i \geq 3$). Then M is a free $R(k)$ -module.*

Proof. By Corollary 3.4, either M is $R(k)$ -free or $M \oplus R(k)$ is $R(k)$ -injective. In the latter case $R(k)$ has to be a Gorenstein ring. However this never occurs as we have already remarked in the proof of the previous corollary. \square

4. MORE RING EXTENSIONS

Let R be an algebra over a field k . And let M be a module over the polynomial ring $R[x]$. The *specialization of M to an element $\alpha \in k$* is defined by

$$M_\alpha := M \otimes_{k[x]} (k[x]/(x - \alpha)k[x]).$$

Remark that if M is a finitely generated $R[x]$ -module, then M_α is a finitely generated R -module.

Lemma 4.1. *Let R be a k -algebra and let $\alpha \in k$. Assume that $x - \alpha$ is a nonzero divisor on $R[x]$ -modules M and N . Then we have an exact sequence*

$$0 \rightarrow \text{Ext}_{R[x]}^i(M, N)_\alpha \rightarrow \text{Ext}_R^i(M_\alpha, N_\alpha) \rightarrow \text{Tor}_1^{k[x]}(\text{Ext}_{R[x]}^{i+1}(M, N), k[x]/(x - \alpha)) \rightarrow 0$$

for each $i \geq 0$.

Proof. From the obvious exact sequence

$$0 \longrightarrow N \xrightarrow{x-\alpha} N \longrightarrow N_\alpha \longrightarrow 0$$

we have a long exact sequence

$$\text{Ext}_{R[x]}^i(M, N) \xrightarrow{f_i} \text{Ext}_{R[x]}^i(M, N) \rightarrow \text{Ext}_{R[x]}^i(M, N_\alpha) \rightarrow \text{Ext}_{R[x]}^{i+1}(M, N) \xrightarrow{f_{i+1}} \text{Ext}_{R[x]}^{i+1}(M, N),$$

where each f_i is a multiplication mapping by $x - \alpha$. Since $x - \alpha$ is a nonzero divisor on M , one can easily see that $\text{Ext}_{R[x]}^i(M, N_\alpha) \cong \text{Ext}_R^i(M_\alpha, N_\alpha)$. Thus it results the exact sequence

$$0 \longrightarrow \text{coker}(f_i) \longrightarrow \text{Ext}_R^i(M_\alpha, N_\alpha) \longrightarrow \ker(f_{i+1}) \longrightarrow 0.$$

By the definition of specialization it is easy to verify that $\text{coker}(f_i) = \text{Ext}_{R[x]}^i(M, N)_\alpha$, and $\ker(f_{i+1}) \cong \text{Tor}_1^{k[x]}(\text{Ext}_{R[x]}^{i+1}(M, N), k[x]/(x - \alpha))$ \square

Related to Conjecture (E) in Section 1, we are now able to the following theorem.

Theorem 4.2. *Suppose that k is an uncountable field and R is a finite dimensional k -algebra of finite Ext-index. Let $k(x)$ be a transcendental extension of k . Then $R \otimes_k k(x)$ is also of finite Ext-index. More precisely, the inequality $\text{Ext-index}(R \otimes_k k(x)) \leq \text{Ext-index}(R)$ holds.*

Proof. Set $b = \text{Ext-index}(R)$ and let M' and N' be finitely generated $R \otimes_k k(x)$ -modules satisfying $\text{Ext}_{R \otimes_k k(x)}^i(M', N') = 0$ for $i \gg 0$. We only have to show that $\text{Ext}_{R \otimes_k k(x)}^i(M', N') = 0$ for $i > b$.

Note that $R \otimes_k k(x)$ is just a localization of $R[x]$ by a multiplicatively closed subset $k[x] \setminus \{0\}$. Hence we can choose a finitely generated $R[x]$ -submodule M of M' (resp. N of N') so that $M \otimes_{k[x]} k(x) \cong M'$ (resp. $N \otimes_{k[x]} k(x) \cong N'$). Notice that $x - \alpha$ acts on M and N as a non-zero divisor for each $\alpha \in k$.

Since we have an isomorphism $\text{Ext}_{R \otimes_k k(x)}^i(M', N') \cong \text{Ext}_{R[x]}^i(M, N) \otimes_{k[x]} k(x)$, we see that $\text{Ext}_{R[x]}^i(M, N) \otimes_{k[x]} k(x) = 0$ for $i \gg 0$.

On the other hand, since R is a finite dimensional k -algebra, each module $\text{Ext}_{R[x]}^i(M, N)$ ($i \geq 0$) is a finitely generated $k[x]$ -module. Hence it has a decomposition as a $k[x]$ -module as follows:

$$\text{Ext}_{R[x]}^i(M, N) \cong \bigoplus_{j=1}^{s_i} k[x]/(f_{ij}(x)) \oplus k[x]^{r_i},$$

where $f_{ij}(x) \neq 0 \in k[x]$.

Since $\text{Ext}_{R[x]}^i(M, N) \otimes_{k[x]} k(x)$ are vanishing for $i \gg 0$, we have $r_i = 0$ for $i \gg 0$. Since there are only countably many equations $f_{ij}(x)$, we can find an element $\alpha \in k$

with the property $f_{ij}(\alpha) \neq 0$ for all i, j . Then, since $x - \alpha$ acts bijectively on $k[x]/(f_{ij}(x))$, we see that $\text{Tor}_1^{k[x]}(\text{Ext}_{R[x]}^{i+1}(M, N), k[x]/(x - \alpha)) = 0$ for all i . And we see as well that $\text{Ext}_{R[x]}^i(M, N)_\alpha = 0$ for $i \gg 0$. Therefore the previous lemma implies that $\text{Ext}_R^i(M_\alpha, N_\alpha) = 0$ for $i \gg 0$. Thus, by the definition of Ext-index, we have $\text{Ext}_R^i(M_\alpha, N_\alpha) = 0$ for all $i > b$. Since $\text{Ext}_{R[x]}^i(M, N)_\alpha$ is a submodule of $\text{Ext}_R^i(M_\alpha, N_\alpha)$ by Lemma 4.1, we have $\text{Ext}_{R[x]}^i(M, N)_\alpha = 0$ for all $i > b$. This implies that $r_i = 0$ for $i > b$, which is equivalent to the vanishing $\text{Ext}_{R[x]}^i(M, N) \otimes_{k[x]} k(x) = 0$ for $i > b$. \square

Remark 4.3. Given an integer $t \geq 1$, suppose there is a natural number n with $\text{Ext}_R^i(M, N) = 0$ for $n + 1 \leq i \leq n + t$ and $\text{Ext}_R^j(M, N) \neq 0$ for $j = n, n + t + 1$. In such a case we say that $\text{Ext}_R(M, N)$ has a gap of length t . Set

$$\text{Ext-gap}(R) := \sup\{t \in \mathbb{N} \mid \text{there are finitely generated } R\text{-modules } M \text{ and } N \text{ such that } \text{Ext}_R(M, N) \text{ has a gap of length } t\}.$$

The ring R is called Ext-bounded if $\text{Ext-gap}(R) < \infty$. We should remark from [7, Theorem 3.4(3)] that if R is a Gorenstein local ring that is Ext-bounded, then R is of finite Ext-index.

Keeping in mind this remark, we can prove the following statement completely in a similar way to the proof of Theorem 4.2:

Theorem 4.4. *Let R be a finite dimensional k -algebra where k is an infinite field, and let $k(x)$ be a transcendental extension of k . If R is Ext-bounded, then so is $R \otimes_k k(x)$. More precisely the inequality $\text{Ext-gap}(R \otimes_k k(x)) \leq \text{Ext-gap}(R)$ holds.*

Proof. Let M' and N' be finitely generated $R \otimes_k k(x)$ -modules and suppose that $\text{Ext}_{R \otimes_k k(x)}(M', N')$ has a gap of length t . To prove $t \leq \text{Ext-gap}(R)$, we use the same notation as in the proof of Theorem 4.2. First we can find finitely generated $R[x]$ -modules M and N and an integer n satisfying $\text{Ext}_{R[x]}^i(M, N) \otimes_{k[x]} k(x) = 0$ for $n + 1 \leq i \leq n + t$ and $\text{Ext}_{R[x]}^j(M, N) \otimes_{k[x]} k(x) \neq 0$ for $j = n, n + t + 1$. As in the proof of Theorem 4.2, we decompose $\text{Ext}_{R[x]}^i(M, N)$ as $k[x]$ -modules into direct sums of indecomposable ones, and we have a finite number of equations $f_{ij}(x)$ ($n + 1 \leq i \leq n + t + 1$, $1 \leq j \leq s_i$). Now we choose an element $\alpha \in k$ that is not a zero of any of these polynomials. Then, as in the same way as the proof of Theorem 4.2, we can show $\text{Ext}_R^i(M_\alpha, N_\alpha) = 0$ for $n + 1 \leq i \leq n + t$. Thus by definition we have $t \leq \text{Ext-gap}R$. \square

Remark 4.5. Let R be a Cohen-Macaulay local ring with dualizing module. As pointed out in [4, Observation (4.1)], by using maximal Cohen-Macaulay approximations, it is easy to see that R is of finite Ext-index if and only if there exists an integer $b \geq 0$, depending only on R , such that $P^R(M, N) \leq b$ for all maximal

Cohen-Macaulay R -modules M and N with $P^R(M, N) < \infty$. In fact, setting

$$\zeta(R) = \sup \{ p^R(M, N) \mid p^R(M, N) < \infty \text{ where } M \text{ and } N \text{ are} \\ \text{maximal Cohen-Macaulay } R\text{-modules} \},$$

we can show

$$\zeta(R) \leq \text{Ext-index}(R) \leq \zeta(R) + d.$$

Therefore R is of finite Ext-index if and only if $\zeta(R) < \infty$.

Note that in the case that R is a Gorenstein local ring, we have the equality $\text{Ext-index}(R) = \zeta(R)$.

Theorem 4.6. *Let (R, \mathfrak{m}, k) be a Cohen-Macaulay local ring of finite Ext-index. Assume that R possesses a dualizing module and that R has an uncountable coefficient field k . Then $R[x]_{\mathfrak{m}R[x]}$ is of finite Ext-index as well. More precisely, the inequality $\zeta(R[x]_{\mathfrak{m}R[x]}) \leq \text{Ext-index}(R)$ holds.*

Proof. It is known and is easily seen that $R[x]_{\mathfrak{m}R[x]}$ is also a Cohen-Macaulay local ring with dualizing module. We only have to show that $\zeta(R[x]_{\mathfrak{m}R[x]}) \leq \text{Ext-index}(R)$.

Let $t = \text{Ext-index}(R)$, and assume that there are maximal Cohen-Macaulay $R[x]_{\mathfrak{m}R[x]}$ -modules M' and N' such that $\text{Ext}_{R[x]_{\mathfrak{m}R[x]}}^i(M', N') = 0$ for $i \gg 0$. We shall show that this vanishing holds for all $i > t$.

As in the proof of Theorem 4.2, we can find finitely generated $R[x]$ -modules M and N , such that $\text{Ext}_{R[x]_{\mathfrak{m}R[x]}}^i(M', N') \cong \text{Ext}_{R[x]}^i(M, N)_{\mathfrak{m}R[x]}$ for all i . For simplicity we denote the $R[x]$ -module $\text{Ext}_{R[x]}^i(M, N)$ by E^i . By Nakayama's lemma, it is easy to see that $E^i_{\mathfrak{m}R[x]} = 0$ is equivalent to that $(E^i/\mathfrak{m}E^i) \otimes_{k[x]} k(x) = 0$.

Note that each $E^i/\mathfrak{m}E^i$ is a finitely generated $k[x]$ -module. Hence, as in the proof of Theorem 4.2, we have the decomposition as $k[x]$ -modules into indecomposable modules:

$$E^i/\mathfrak{m}E^i \cong \bigoplus_{j=1}^{s_i} k[x]/(f_{ij}(x)) \oplus k[x]^{r_i},$$

where $f_{ij} \neq 0$ are irreducible polynomials of $k[x]$. As before we can choose an element $\alpha \in k$ so that $f_{ij}(\alpha) \neq 0$ for all i, j . Since $x - \alpha$ acts bijectively on $k[x]/(f_{ij}(x))$, we see that $(E^i/\mathfrak{m}E^i)_{\alpha} = k^{r_i}$ for i . By the assumption, since $r_i = 0$ holds for $i \gg 0$, we have $(E^i/\mathfrak{m}E^i)_{\alpha} = 0$ for $i \gg 0$.

Note that $(E^i/\mathfrak{m}E^i)_{\alpha} \cong E_{\alpha}^i/\mathfrak{m}E_{\alpha}^i$ holds by a trivial reason. Thus it follows from Nakayama's lemma that $E_{\alpha}^i = 0$ for $i \gg 0$. This means that $E^i \xrightarrow{x-\alpha} E^i$ is bijective for $i \gg 0$. Therefore we have $\text{Tor}_1^{k[x]}(E^{i+1}, k[x]/(x - \alpha)) = 0$ for $i \gg 0$.

Hence by Lemma 4.1, $\text{Ext}_R^i(M_{\alpha}, N_{\alpha}) = 0$ for $i \gg 0$, and thus $\text{Ext}_R^i(M_{\alpha}, N_{\alpha}) = 0$ for $i > t$, by the definition of t . Use Lemma 4.1, and we have that $E_{\alpha}^i = 0$ for all $i > t$. In particular, $(E^i/\mathfrak{m}E^i)_{\alpha} = 0$ for $i > t$. This means that $r_i = 0$ for $i > t$ and hence $E^i/\mathfrak{m}E^i \otimes_{k[x]} k(x) = 0$ for $i > t$, equivalently $E_{\mathfrak{m}R[x]}^i = 0$ for $i > t$. This shows $\zeta(R[x]_{\mathfrak{m}R[x]}) \leq t$ as desired. \square

Remark 4.7. As in Remark 4.3, we can prove the following statement completely in the same way as the proof above:

Let (R, \mathfrak{m}, k) be an Ext-bounded Cohen-Macaulay local ring admitting dualizing module. Assume that R has an infinite coefficient field k . Then $R[x]_{\mathfrak{m}R[x]}$ is also Ext-bounded.

ACKNOWLEDGMENTS

The whole work of this paper was done during the visit of the first author to Department of Mathematics of Okayama University. The first author is grateful to Okayama University for its hospitality and facilities.

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